MEASURE OF INCONSISTENCY FOR POTENTIAL METHOD

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ABSTRACT. Inconsistency of decision maker's preferences may be measured as a number of violation of a transitivity rule or by some other measure like consistency index in Analytical Hierarchy Process. In Potential Method, inconsistency is measured by an angle between the preference flow and the column space of the incidence matrix. In this article, a random study is performed to determine the upper bound for admissible inconsistency. The degree distribution is recognized as a Gumbel distribution and the upper bound for admissible inconsistency measure is defined as a p-quantile ($p = 0.05$) of that distribution.

1. INTRODUCTION

Inconsistency measure is a useful information which shows a degree of non-transitivity in decision maker's preferences. The high inconsistency measure may suggests her/him to reconsider the input again and again if necessary. For the Eigenvalue Method (EVM), proposed by Saaty [2], a Consistency Index CI is defined as

$$\text{CI}(A) = \frac{\lambda_{\max}(A) - n}{n-1},$$

where A is the positive reciprocal matrix of order n and $\lambda_{\max}(A)$ is the Perron root of A. It is well known that $\text{CI} \geq 0$ and $\text{CI}(A) = 0 \iff A$ is consistent. A positive reciprocal matrix A is of *admissible inconsistency* if

$$\text{CI}(A) \leq 0.1 \times \text{MRI}(n) \tag{1}$$

where $\text{MRI}(n)$ is the mean of the random CI. The random index study in AHP context was performed by several authors, from Crawford and Williams [6] (1985) until Alonso and Lamata [7] (2006). A nice overview of the results is given in the [7, Table 1, p. 449].

The Potential Method (PM) uses a *preference graph* to capture the results of pairwise comparisons. A *preference flow* $\mathcal{F}$, defined on the set of arcs (preferences), is the intensity of the preference. The aim of this article is to perform a random study to define the upper bound for admissible inconsistency of the flow $\mathcal{F}$ inspired by the random index study for EVM.

In the Section 2 we introduce a notation, develop the idea of consistency and explain a connection with other techniques (reciprocal matrix, stochastic preference). The basic formula is the Laplace equation $A^\tau AX = A^\tau \mathcal{F}$ which has to be solved if $\mathcal{F}$ is not consistent, A denotes the incidence matrix of the preference graph.

The Section 3 describes a connection of PM with Geometric Mean, Ordinal value function and Stochastic preference model.

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In the Section 4 we perform a randomization procedure to determine the admissible level of the flow inconsistency. Analytical Hierarchy Process (AHP), more precisely the Eigenvalue Method (EVM), serve as a model. Several randomization procedure were examined to simulate decision maker's preferences. It is shown that the empirical distribution of the inconsistency measure DEG may be modeled as a Gumbel distribution. The upper bound for admissible inconsistency is defined as the 0.05-quantile of the theoretical distribution.

In the Section 5 a Condition of Order Preservation (COP) is considered and the number of violation #VCOP is calculated for the random graph and the random reciprocal matrix. It is shown that the Consistency Index (CI) is not correlated with #VCOP while the correlation between the random degree DEG and #VCOP is very good.

2. Preference graph

2.1. **Consistent preference flow.** A *preference graph* is a digraph $\mathcal{G} = (V, \mathcal{A})$ where V is the set of nodes and $\mathcal{A}$ is the set of arcs of $\mathcal{G}$. We say that the node a is *more preferred* than node b, in notation $a \succcurlyeq b$, if there is an arc (a, b) outgoing from b and ingoing to a. A *preference flow* is a non-negative real function $\mathcal{F}$ defined on the set of arcs. The value $\mathcal{F}_\alpha$ on the arc α is the intensity of the preference on some scale[1]. For the arc $\alpha = (a, b)$, $\mathcal{F}_\alpha = 0$ means that the decision maker is indifferent for the pair $\{a, b\}$. In that case the orientation of the arc is arbitrary.

Incidence matrix $A = (a_{\alpha,v})$ of the graph is defined as an $m \times n$ matrix, $m = \operatorname{Card} \mathcal{A}, n = \operatorname{Card} V$ where

$$a_{\alpha,v} = \begin{cases} -1, & \text{if the arc } \alpha \text{ leaves the node } v \\ 1, & \text{if the arc } \alpha \text{ enters the node } v \\ 0, & \text{otherwise.} \end{cases}$$

We shall write a_{ij} where i is the index of i-th arc and j is the index of j-th node. The vector space $\mathbb{R}^m$ is called *arcs space* and the vector space $\mathbb{R}^n$ is called *vertex space*. The incidence matrix (and matrix in general) generates an orthogonal decomposition

$$N(A^\tau) \oplus R(A) = \mathbb{R}^m$$

where $R(A)$ is the column space of the matrix A and $N(A^\tau)$ is the null-space of the matrix A^τ. $N(A^\tau)$ is called the *cycle space* because it is generated by all cycles of the graph.

Definition 1. *A preference flow $\mathcal{F}$ is consistent if there is no component of that flow in the cycle-space.*

According to the definition, $\mathcal{F}$ is consistent if the sum of algebraic components of the flow along each cycle is equal to zero. Evidently,

Theorem 2. *The following statements are equivalent:*

(1) $\mathcal{F}$ *is* consistent.
(2) $\mathcal{F}$ *is a linear combination of the columns of the incidence matrix A.*
(3) *There exists $X \in \mathbb{R}^n$ such that $AX = \mathcal{F}$.*
(4) *The scalar product $y^\tau \mathcal{F} = 0$ for each cycle y, i.e. $\mathcal{F}$ is orthogonal to the cycle space.*

[1] For subjective pairwise comparisons the scale is $\{0, 1, 2, 3, 4\}$.

We may test the consistency of the given flow $\mathcal{F}$ by solving the equation

$$AX = \mathcal{F}. \tag{2}$$

Definition 3. *A solution of the equation $AX = \mathcal{F}$, if it exists, is called the potential of $\mathcal{F}$.*

Evidently, X is not unique because the constant column $\mathbb{1}^\tau = [\, 1 \ \cdots \ 1 \,]^\tau$ is an element of the kernel $N(A)$. The equation (2) may be rewritten as

$$\mathcal{F}_\alpha = X(a) - X(b) \text{ for } \alpha = (a,b) \in \mathcal{A} \tag{3}$$

which means that X is a measurable value function i.e. it measures the preference on the interval scale. For the consistent flow it is easy to find a potential X using a spanning tree of the preference graph (if it is connected). The details are left to the reader.

An example of the preference graph in a voting procedure was considered by [1, Condorcet] (1785). He defined a *social preference* flow as

$$\mathcal{F}_C(u,v) := N(u,v) - N(v,u)$$

where $N(u,v)$ denotes the number of voters choosing u over v. We say that u is *socially preferred* to v if $\mathcal{F}_C(u,v) \geq 0$. It is easy to see that the Condorcet's winner exists if the Condorcet's graph has no positive cycle.

2.1.1. *A comment on consistency.* There is a notion of the consistent matrix in Saaty's AHP method [2] and the theorem which states that a positive reciprocal matrix $A = (a_{ij})$ is consistent if and only if

$$a_{ij}a_{jk} = a_{ik}, \quad i,j,k = 1,\ldots,n \tag{4}$$

Taking a logarithm of this relation one can recognize the condition (4) from Theorem 2 for the flow

$$\mathcal{F}_{(i,j)} := \log(a_{ij}). \tag{5}$$

In the stochastic preference approach, French [3, p. 101], the author introduces a notion of the stochastic preference p_{ab} as a probability of choosing a when offered a choice between a and b. Then, it is easy to show that if the stochastic preference satisfies the consistency condition

$$\frac{p_{ab}}{p_{ba}} \cdot \frac{p_{bc}}{p_{cb}} = \frac{p_{ac}}{p_{ca}} \tag{6}$$

for all $a,b,c \in V$ then, it induces a weak preference order and generates an ordinal value function. If we define a *stochastic flow* $\mathcal{F}$ by

$$\mathcal{F}_{(b,c)} := \log \frac{p_{bc}}{p_{cb}},$$

then, the stochastic preference is consistent if and only if the flow $\mathcal{F}$ is consistent in the sense of (2). This shows that the consistency of the flow has its counterparts in different approaches.

2.2. **Inconsistent flow.** In practice, a decision maker, while performing pairwise comparisons, does not give the flow which is necessarily consistent. The best approximation of that flow by the column space of the incidence matrix may be calculated in this situation. Its *potential* X is a solution of the normal equation (or Laplace equation) of (2), i.e.

$$A^\tau AX = A^\tau \mathcal{F}. \tag{7}$$

For uniqueness, we pose the condition

$$\sum_{v\in V} X(v) = 0. \tag{8}$$

It is easy to prove the following theorem (the proof is left to the reader):

Theorem 4. *If the preference graph is complete, i.e. each two nodes are adjacent, than the value of the potential X at the node v node is*

$$X(v) = \frac{1}{n}\Big(\sum_{\alpha\in \mathrm{In}(v)} \mathcal{F}_\alpha - \sum_{\alpha\in \mathrm{Out}(v)} \mathcal{F}_\alpha\Big), \tag{9}$$

where $\mathrm{Out}(v)$ *and* $\mathrm{In}(v)$ *denote the set of all outgoing and ingoing arcs for v.*

We may simplify the formula (9) by introducing the *flow matrix* F

$$F_{ij} = \begin{cases} \mathcal{F}_{(i,j)} & \text{if } (i,j)\in\mathcal{A}, \\ -\mathcal{F}_{(j,i)} & \text{if } (j,i)\in\mathcal{A}, \end{cases}$$

with the convention $F_{ii} = 0$. The matrix F is antisymmetric and the potential X, defined by (9), is the arithmetic mean of the columns of F, i.e.

$$x_i = \frac{1}{n}\sum_{j=1}^{n} F_{ij},\ i = 1,\dots,n. \tag{10}$$

2.3. **Aggregation of flows.** Let us suppose that a group of decision makers (DM) is finite and each DM have his own preference graph over the set of alternatives (the same set for all DMs). In the hierarchical model, each member of the group may use his own hierarchy with his own criteria.

2.3.1. *Consensus graph (flow).* A procedure which we are going to describe is the same if instead of a group of decision makers we have two or more criteria/goals.

Let us suppose that each criterion $C_i, i\in\{1,\dots,k\}$ has its own preference graph $(V,\mathcal{A}_i)$ and its own preference flow $\mathcal{F}_i$. If w_i denotes the weight of i-th criterion then, for a given pair $\alpha = (u,v)$ of the alternatives we calculate

$$F_\alpha := \sum_{\substack{i=1\\ \pm\alpha\in\mathcal{A}_i}}^{k} w_i\mathcal{F}_i(\alpha) \tag{11}$$

where the item $w_i\mathcal{F}_i(\alpha)$ is taken into account if and only if $\alpha\in\mathcal{A}_i$ or $-\alpha\in\mathcal{A}_i$. If this sum is non-negative, then we include α in the set $\mathcal{A}$ of arcs of the *consensus graph*, and we put $\mathcal{F}(\alpha) := F_\alpha$. If the sum is negative, we define $-\alpha = (v,u)$ as an arc in $\mathcal{A}$ and $\mathcal{F}(-\alpha) := -F_\alpha$. If F_α is not defined then u and v are not adjacent in the consensus graph.

2.3.2. *Hierarchical data structure.* In the hierarchical decision structure each node, except the nodes in the last level, is a parent node for its children from some other level. A parent node may be considered as the criterion for children evaluation. The only restriction is that all children of the parent should be in the same level while the parents of the node may be from different levels. A restriction, made by the conservation law, is that the sum of the weights of the nodes in some level set should be the sum of the weights of their parents.

PM calculates the weights of nodes in some level in the following way. First, the weight of the goal is set to be 1. For a particular level which is not yet ranked, the aggregation of flows is made over the set of all parents, potential X is calculated after that the weights w are obtained using the formula

$$w = k \cdot \frac{a^X}{\|a^X\|_1} \tag{12}$$

where $\|\cdot\|_1$ represents l_1-norm and k is the sum of weights of the parents (usualy k=1). The process is repeated until the bottom level of the hierarchy is ranked.

The exponential function $X \mapsto a^X$ is defined by the components and $a > 1$ is a positive constant. Currently, we use the value $a = 2$ but user may precise some other value. If a preference graph is not connected the above procedure should be done for each connected component.

2.3.3. *Trading off between criteria.* In MCDM the *trading off* between criteria may be performed if criteria use different measurement scales. This may be done by renormalization of the flow components in such a way that the maximal component of the flow has a given value called the *flow norm*.

3. Potential and other methods

3.1. **Potential and geometric mean.** In AHP the results of pairwise comparisons are measured on ratio scale and stored in a positive reciprocal matrix A. The logarithm of A, taken by components,

$$F_{ij} = \log_a a_{ij}, \quad a > 0$$

is an antisymmetric matrix F which is the flow matrix of some flow $\mathcal{F}$. The potential X of $\mathcal{F}$ may be expressed in terms of the matrix A, using the formula (10), as

$$x_i = \frac{1}{n}\sum_j F_{ij} = \frac{1}{n}\sum_j \log_a a_{ij} = \log_a \Big(\prod_j a_{ij}\Big)^{\frac{1}{n}},$$

and the weight w_i, using (12), can be written as the row geometric mean

$$w_i = \Big(\prod_j a_{ij}\Big)^{\frac{1}{n}},\ i = 1,\dots,n.$$

3.2. **Potential as ordinal value function.** Suppose, for the moment, that $\mathcal{F}$ is an uni-modular flow, i.e. $\mathcal{F}_\alpha \in \{0,1\}, \forall\alpha \in \mathcal{A}$. In that case, we may define the relation

$$u \succcurlyeq v \Leftrightarrow \mathcal{F}_{(u,v)} \geq 0.$$

If $\succcurlyeq$ is a weak preference relation then, a natural question is whether the potential X is an ordinal value function. The following theorem is more precise.

Theorem 5. *If the flow $\mathcal{F}$ is uni-modular and $\succcurlyeq$ is a weak preference relation then, the potential X is an ordinal value function and*

$$\mathcal{F}_{(a,b)} \geq 0 \Leftrightarrow X(a) - X(b) \geq 0.$$

The proof can be found in Čaklović [5].

3.3. **Potential of the stochastic flow.**

For the complete stochastic flow defined by the formula (6) we may calculate the potential X by formula (10).

$$X(a) = \frac{1}{n}\sum_{b\neq a} F_{ab} = \frac{1}{n}\sum_{b\neq a}(\log p_{ab} - \log p_{ba}) = \frac{1}{n}\sum_{b\neq a}\log\frac{p_{ab}}{p_{ba}} = \log\left(\prod_{b\neq a}\frac{p_{ab}}{p_{ba}}\right)^{\frac{1}{n}}. \tag{13}$$

and the weight of the node a is, by formula (**??**),

$$w_a = \left(\prod_{b\neq a}\frac{p_{ab}}{p_{ba}}\right)^{\frac{1}{n}}. \tag{14}$$

4. Admissible inconsistency

Measure of inconsistency of the flow $\mathcal{F}$, in notation $\text{DEG}(\mathcal{F})$, is defined as the angle between $\mathcal{F}$ and the column space of the incidence matrix, i.e. the angle between $\mathcal{F}$ and AX where AX is the consistent approximation of $\mathcal{F}$. Evidently, $\text{DEG}(\mathcal{F}) = 0$ if and only if $\mathcal{F}$ is consistent.

In this section we shall determine the distribution of the inconsistency measure of the random flow. For $n \geq 4$ this distribution is recognized as Gumbel distribution

$$e(x) = \frac{e^{-e^{\frac{-x+\alpha}{\beta}}+\frac{-x+\alpha}{\beta}}}{\beta}, \tag{15}$$

which parameters (slightly) depend upon the randomization procedure of the preference flow, see Table 4. For instance, if the randomization is made as a log-normal perturbation of the random consistent flow as in formula (16), the inconsistency measure is Gumbel Distribution $E(\alpha = 17.61, \beta = 7.03)$ (Figure 4).

4.1. **Randomization.**

Random index study in AHP context was performed by several authors, from Crawford and Williams [6] (1985) until Alonso and Lamata [7] (2006). A nice overview of the results is given in [7, Table 1, p. 449].

A randomization of the preference can be designed, generally speaking, as: random *perturbation* and random *distribution* of the preference. The randomization which we perform here are:

(1) normal perturbation of the consistent flow (reciprocal matrix),

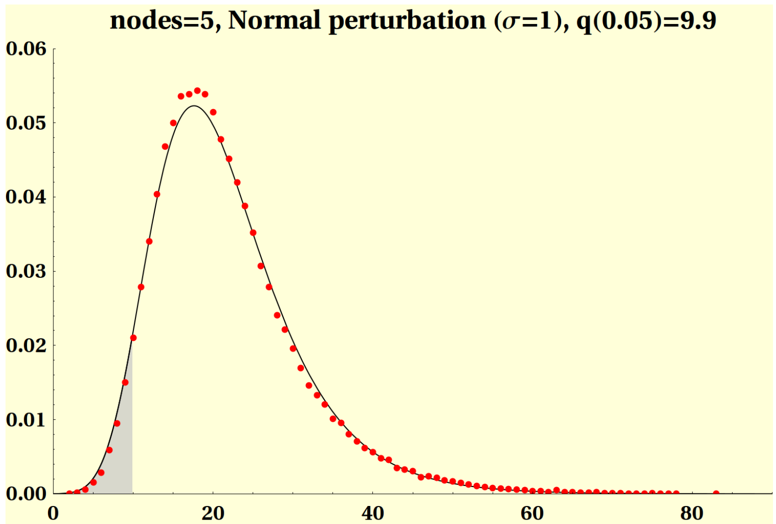

FIGURE 1. Simulated distribution of the inconsistency measure (dots). Number of nodes equals 5. Number of simulations is 10^5. 0.05-quantile (9.9) is taken as an upper bound for admissible inconsistency.

(2) uniform perturbation of the consistent flow (reciprocal matrix),

Random distribution (instead of random perturbation) is not appropriate because it generates highly inconsistent flows with average greater than 50^o.

In AHP context we performed the randomization using distribution with standard deviation $\sigma = 1$ and our results are exactly those of Noble [8]. A random positive reciprocal matrix is obtained as the normal perturbation of the random consistent reciprocal matrix with elements

$$a_{ij} = w_{ij} * \exp(N(0, \sigma)) \tag{16}$$

where $\sigma = 1$ and $w_{ij} := int(rand(1\text{-}9))^{\alpha}$ $(i < j)$, is the random choice from the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, powered by α which is the random choice from $\{-1, 1\}$, and $w_{ij} := w_{ji}^{-1}$, for $j < i$, and $w_{ii} = 1$, $\forall i = 1, \ldots, n$.

Random consistent flow is made by random choice of the orientation of the arc and by random choice of the flow value in the set $\{0, 1, 2, 3, 4\}$. Normal perturbation has standard deviation $\sigma = 1$.

Randomization procedure was done by `Perl` and data analysis was done by `R`.

4.2. **Admissible inconsistency.** It seems reasonable to determine the upper bound for admissible inconsistency as a p-quantile of the theoretical random degree distribution. Those values for $p = 0.05$ are given in table 4 in the column *'theoretical'*. Quantiles of the experimental data are given in the column *'from data'*.

If the number of vertices in the preference graph is $n = 3$ the distribution is not a Gumbel distribution. A reason may be in the restriction on the stochastic flow values, i.e $\mathcal{F}_\alpha \in \{0, 1, 2, 3, 4\}$. A p-quantile in this case should be recalculated in a slightly different way, may be with less restrictions.

		Gumbel Distribution $E(\alpha,\beta)$			
number of nodes	perturbation ($\sigma=1$)	0.05-quantile		Parameters	
		from data	theoretical	α	β
3	normal	1	-3.1	9.43	11.41
	uniform	1	-2.8	9.81	11.48
4	normal	6	5.3	15.01	8.83
	uniform	7	5.9	15.43	8.66
5	normal	10	9.9	17.61	7.03
	uniform	11	10.6	18.09	6.81
6	normal	13	12.7	19.18	5.91
	uniform	13	13.4	19.59	5.6
7	normal	15	14.7	20.24	5.07
	uniform	15	15.3	20.59	4.80
8	normal	16	16.1	21.03	4.47
	uniform	17	16.7	21.34	4.22
9	normal	17	17.2	21.64	4.02
	uniform	18	17.7	21.88	3.79
10	normal	18	18.0	22.06	3.67
	uniform	18	18.5	22.31	3.44
11	normal	18	18.8	22.49	3.38
	uniform	16	16.	19.02	2.72
12	normal	19	19.3	22.77	3.16
	uniform	16	16.5	19.24	2.54
13	normal	19	19.8	23.04	2.96
	uniform	17	16.8	19.46	2.38
14	normal	20	20.2	23.28	2.79
	uniform				
15	normal	20	20.6	23.47	2.66
	uniform	17	17.4	19.79	2.13

TABLE 1. Quantiles of random degree as a function of the nodes number (10^5 simulations).

5. Condition of Order Preservation

We say that X satisfies the Condition of Order Preservation (COP) if

$$\mathcal{F}_{ij} > 0 \;\&\; \mathcal{F}_{kl} > 0 \text{ and} \tag{17}$$

$$\mathcal{F}_{(i,j)} > \mathcal{F}_{(k,l)} \implies X_i - X_j > X_k - X_l. \tag{18}$$

In contrast to measure of inconsistency DEG($\mathcal{F}$) which is an 'a priori' inconsistency measure, the number of violation of COP may be regarded as an 'a posteriori' measure of inconsistency which shows 'how far' the calculated potential X is from a measurable value function.

For a reciprocal positive matrix A, we say that COP is satisfied if

$$a_{ij} > 0 \ \& \ a_{kl} > 0 \text{ and}$$
$$a_{ij} > a_{kl} \implies \frac{w_i}{w_j} > \frac{w_k}{w_l},$$

where w is the Perron eigenvector of A.

In this section we present the results of a statistical comparison of a number of violation of Condition of Order Preservation (#VCOP) between EVM and PM. For this purpose we performed 10^4 simulations of 5×5 positive reciprocal matrix. For each randomly generated reciprocal matrix we calculate its consistency index CI and #VCOP(EVM) for EVM. Then, we calculate the measure of inconsistency DEG($\mathcal{F}$) of the flow $\mathcal{F}$ which (the logarithm of the reciprocal matrix) and #VCOP(PM) generated by PM. The correlation matrix of the random vector (DEG, #VCOP(PM), CI, #VCOP(EVM)) is given in the Table 2:

	DEG	#VCOP(PM)	CI	#VCOP(EVM)
DEG	1.	0.811266	0.55142	0.817288
#VCOP(PM)		1.	0.449875	0.950875
CI			1.	0.460128
#VCOP(EVM)				1.

TABLE 2. Correlation matrix for the random vector (DEG, #VCOP(PM), CI, #VCOP(EVM)).

At first glance it is evident that the inconsistency measure DEG of the preference flow better predicts the #VCOP than the consistency index. What does it mean from the user point of view we cannot say at this moment. The #VCOP is some quantitative information about the obtained ranking and it seems reasonable to inform a decision maker about this number together with inconsistency measure of the input data. We do not impose the zero value of #VCOP as a standard, we just want to say that #VCOP gives some new information about inconsistency from the point of view of metric topology.

6. CONCLUSION

This article speaks about inconsistency of the preference flow and the upper bound for admissible inconsistency. The upper bound is determined as a p-quantile of the theoretical distribution which is recognized as the Gumbel distribution.

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