MEASURE OF INCONSISTENCY. AHP & POTENTIAL MEETHOD. A COMPARISON.

LAVOSLAV ČAKLOVIĆ

Preference graph

A ***preference graph*** is a digraph $\mathcal{G} = (V, \mathcal{A})$ where V is a set of nodes and $\mathcal{A}$ is a set of arcs of $\mathcal{G}$. We say that the node a is ***more preferred*** than the node b, in notation $a \succcurlyeq b$, if there is an arc (a, b) outgoing from b and ingoing to a. A ***preference flow*** is a non-negative real function $\mathcal{F}$ defined on the set of arcs. The value $\mathcal{F}_\alpha$ on the arc α is an intensity of the preference on some scale[1]. For the arc $\alpha = (a, b)$, $\mathcal{F}_\alpha = 0$ means that the decision maker is indifferent for the pair $\{a, b\}$. In that case orientation of the arc is arbitrary. A preference flow is ***consistent*** if there is no component of the flow in the cycle-space of the graph. According to that definition, $\mathcal{F}$ is consistent if the sum of algebraic components of the flow along each cycle is equal to zero. Equivalently, $\mathcal{F}$ is consistent if there exists $X \in \mathbb{R}^n$ (called ***potential***) such that $AX = \mathcal{F}$, where A is an incidence matrix of the graph. If $\mathcal{F}$ is not consistent, the potential X may be calculated as a solution of the equation $AX = \mathcal{F}_0$ where $\mathcal{F}_0$ is the best approximation of $\mathcal{F}$ in the column space of the matrix A. It is evident that a potential X of the consistent flow is a measurable value function on the set of nodes, i.e.

$$\mathcal{F}_{(a,b)} \geq 0 \iff X(a) \geq X(b)$$
$$\mathcal{F}_{(a,b)} \geq \mathcal{F}_{(c,d)} \iff X(a) - X(b) \geq X(c) - X(d).$$

In Saaty's Eigenvalue Method (EVM) input data are captured in a positive reciprocal matrix $A = (a_{ij})$. By definition, A is consistent if

$$a_{ij}a_{jk} = a_{ik}, \;\; i, j, k = 1, \ldots, n.$$

A connection between those two types of consistency is the following: If we define a flow by

$$\mathcal{F}_{(i,j)} := \log(a_{ij}).$$

then, $\mathcal{F}$ is consistent iff A is consistent.

Rank reversal and Randomization.

We say that a ***condition of order preservation*** (COP) for AHP is satisfied if

$$a_{ij} > a_{kl} \implies \frac{w_i}{w_j} > \frac{w_k}{w_l},$$

where $A = (a_{ij})$ is the reciprocal positive matrix and w its Perron eigenvector. For the preference flow $\mathcal{F}$, we say that COP is satisfied if:

$$\mathcal{F}_{(i,j)} > \mathcal{F}_{(k,l)} \implies X_i - X_j > X_k - X_l.$$

[1]For subjective pairwise comparisons the scale is $\{0, 1, 2, 3, 4\}$.

A measure of inconsistency in EVM is given by ***inconsistency index*** (CI), in Potential Method (PM) inconsistency is measured by an ***angle*** between $\mathcal{F}$ and $\mathcal{F}_0$ measured in degrees (DEG). To compare those measures we performed 10^4 simulations of positive reciprocal matrix. For each randomly generated reciprocal matrix we calculate its inconsistency index CI the inconsistency measure DEG of the corresponding flow and a number of violations of COP (NOV) for EVM and PM respectively.

CONCLUSION.

It is shown that NOV and DEG are very good correlated for PM ($r = 0.811$) and NOV and CI are not good correlated for EVM ($r = 0.460$) and NOV for both methods are highly correlated ($r = 0.951$), which allows to conclude that inconsistency measure for EVM is not well designed and it should be used by caution. It is also shown that DEG is distributed as a Gumbel Distribution. For instance, if the randomization is made as a log-normal perturbation $N(0, 1)$ of the random consistent flow, the inconsistency measure DEG is distributed as the Gumbel Distribution $E(\alpha = 17.61, \beta = 7.03)$. This allows to define an ***upper bound*** for admissible inconsistency of DEG as a p-quantile ($p = 0.05$) of the random degree distribution as a function of the number of nodes in the graph.

TABLE 1. Quantiles of random degree as a function of the nodes number. 10^5 simulations.

		Gumbel Distribution $E(\alpha, \beta)$			
nodes number	perturbation ($\sigma = 1$)	0.05-quantile from data	0.05-quantile theoretical	α	β
3	normal	1	-3.1	9.43	11.41
	uniform	1	-2.8	9.81	11.48
4	normal	6	5.3	15.01	8.83
	uniform	7	5.9	15.43	8.66
5	normal	10	9.9	17.61	7.03
	uniform	11	10.6	18.09	6.81
6	normal	13	12.7	19.18	5.91
	uniform	13	13.4	19.59	5.6
7	normal	15	14.7	20.24	5.07
	uniform	15	15.3	20.59	4.80
8	normal	16	16.1	21.03	4.47
	uniform	17	16.7	21.34	4.22
9	normal	17	17.2	21.64	4.02
	uniform	18	17.7	21.88	3.79
10	normal	18	18.0	22.06	3.67
	uniform	18	18.5	22.31	3.44

In randomization procedure we used *Perl* and data anlysis was performed by *Mathematica* and *R*.

FACULTY OF NATURAL SCIENCES, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ZAGREB, BIJENIČKA 30, 10 000 ZAGREB, CROATIA, CAKLOVIC@MATH.HR